# Higher Order Natural Element Method – Boundary Element Method Coupling

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**The advantage of a hybrid approach coupling natural element method (NEM) and boundary element method (BEM) is the conjunction of the inherent NEM accuracy and the BEM ability in modeling linear and deformable domains without any mesh. In a previous work, the coupling between standard NEM (1st order consistency) and a 0th order BEM was introduced. In the present work the performance of this coupling is tested for both methods in higher order versions. Results are presented in terms of convergence analysis and compared to a classic FEM-BEM coupling in terms of accuracy. The approach developed yields more accurate results with better convergence behavior, proving to be a good alternative in terms of accuracy for the simulation of unbounded problems.**

***Index Terms*—boundary element method, coupled numerical methods, natural element method.**

## I. Introduction

Air regions in computational electromagnetics are almost ubiquitous. In magnetostatics, for example, the accuracy of field quantities in air gaps is essential. In this context the Boundary Element Method (BEM) takes advantage over others methods, once it can model linear and deformable domains requiring only the discretization of theirs interfaces [1].

Others regions, like magnetic materials, must be modeled by other numerical methods. Traditionally Finite Element Method (FEM) is used in these domains. However, in this work, the Natural Element Method (NEM) [2] is employed. It has been previously shown that this method can provide highly accurate and computationally efficient solutions compared to FEM [3].

In a previous work [4] the coupling NEM-BEM was first introduced. In that paper, the scheme was based on a 1st order consistent NEM and a 0th order BEM. For the present work the consistency of both methods was improved to the 2nd order. The performance of this improved scheme is tested and compared to the previous one. The tests presented in this abstract have been performed on a 2D magnetostatic problem.

## II. Scheme Foundations

In this section we present briefly the main concepts involved on the implementation of the higher order NEM-BEM coupling.

### A. Natural Element Method

NEM standard shape functions are evaluated through geometrical relations based on a discretization scheme called *Voronoï diagram* [2]. At the same time that these shape functions provide smooth interpolation they keep important properties, like partition of unity, positiveness, and strict interpolation [2]. Among other things, these properties allow easy FEM-like implementation of resolution algorithms. However, as NEM shape functions present originally linear consistency, some additional technique must be applied in order to get higher order approximations. This issue was addressed in [5], when a higher order NEM method based on the *de Boor* algorithm was developed. The main idea of the technique is the combination of different levels of linear interpolation that generates approximations with controllable consistency and precision. The method was first applied to the electromagnetic domain in [6] with good results.

### B. Boundary Element Method

If no source field in the free space is taken into account, the problem can be represented by a Laplacian equation in terms of a total scalar magnetic potential. The main idea is to solve Laplacian equation transforming the volume integral equation over the deformable domain into a surface integral over the boundary. The classic boundary integral equation is obtained by using the third Green's identity [1] and solved using the point collocation method. The scalar potential and the normal component of the magnetic induction are both interpolated using 2nd order functions.

### C. Coupling both methods

The results presented here are obtained through a reduced magnetic scalar potential. The coupling between NEM and BEM systems is taken into account by imposing the conservation of the normal component of the magnetic induction and the unicity of the potential on boundary. It is important to notice that the BEM part of the matrix is fully populated and the NEM part is sparse. By solving the coupled formulation, we get the reduced scalar magnetic potential on all nodes and the normal component of the magnetic induction on the boundary.

## III. Test Case: U Shaped Electromagnet

### A. Model

Fig. 1 shows a U shaped electromagnet. In order to test the proposed scheme the magnetic induction was computed on the path 1, in the air. The reduced magnetic field was computed on the second path, inside the electromagnet.

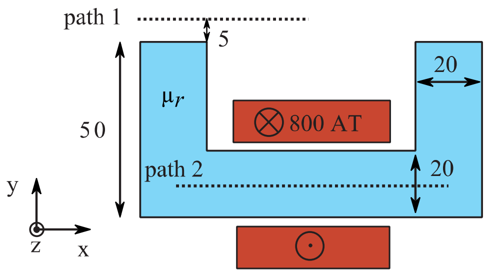


Fig. 1. U shaped electromagnet. Material with linear permeability and a non-meshed coil. Two different paths are considered.

### A. Interpolating the total magnetic field in the unbounded domain

Fig. 2 presents the comparison between the new formulation and the 2nd order FEM solution (reference) for the evaluation of the magnetic induction vertical. This reference was obtained through a mesh with more than 400,000 degrees of freedom (DoF).

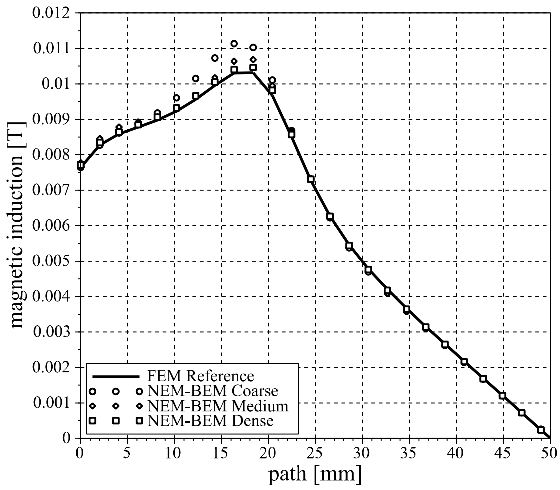


Fig. 2. Vertical component of the magnetic induction on path 1: comparison between FEM 2nd order and the approach presented in this work. The number of DoF for each NEM-BEM simulation case can be consulted in the Fig. 3.

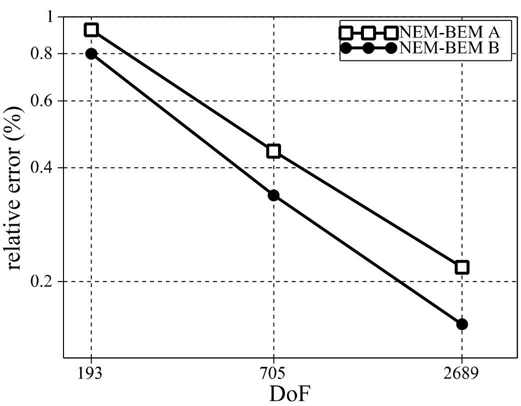


Fig. 3. Relative error ($L_2$ norm) versus the number of DoF (three mesh densities: coarse, medium and dense).

Fig. 3 presents the relative error on the vertical component of the magnetic induction for both NEM-BEM A (results from the previous work [4]) and NEM-BEM B (results from this new formulation) approaches.

### B. Interpolating the magnetic induction in the magnetic material

Fig. 4 presents the comparison between the tested methods in terms of the horizontal component of the reduced magnetic field. It can be seen that the new scheme is smoother and closer to the reference if it is compared with the previous NEM-BEM and FEM-BEM approaches addressed in [4].

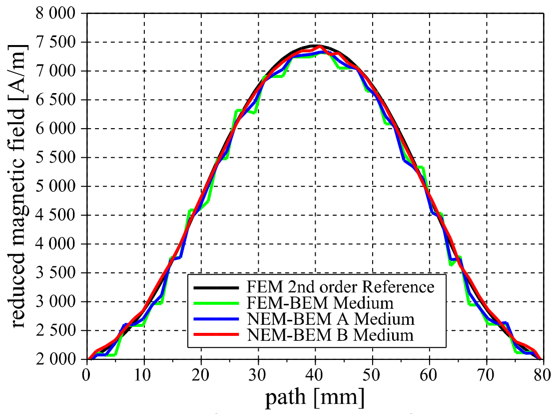


Fig. 4. Horizontal component of the reduced magnetic field on path 2: comparison between the reference and different hybrid approaches.

## IV. Conclusion

Since coupled approaches using BEM are still currently being applied [7], we propose an improvement of these schemes. In this paper, a hybrid NEM-BEM 2nd order formulation has been presented. It was able to provide accurate solutions even with very coarse discretizations. Regarding the next steps, we plan to compare the performance of this new formulation with the FEM using Whitney elements of 2nd degree. We also intend to compare the results presented in this paper, using a reduced magnetic scalar potential, with those from the coupled reduced and total scalar potentials or $\mathbf{t_0}$-$\varphi$ formulations [8].

## V. References


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